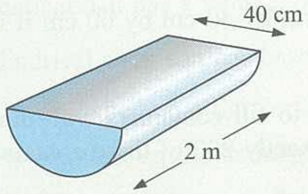
4 A horse's drinking trough has the dimensions shown. How long will it take to fill the trough if water flows into it at 12 litres per minute?



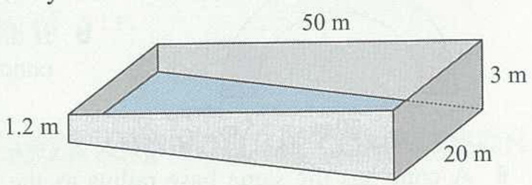
5 A petrol can is a rectangular prism with base measurements 15 cm by 30 cm. If the can has capacity 18 litres, find its height.

6 A concrete driveway is to be 28 m long, 3 m wide, and have an average depth of 12 cm.
How much will it cost to lay the driveway if the contractor charges €160 per cubic metre of concrete?

7 A 50 m long swimming pool is completely filled with water. Find:

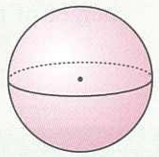
a the area of the trapezium side

b the capacity of the pool in kL.



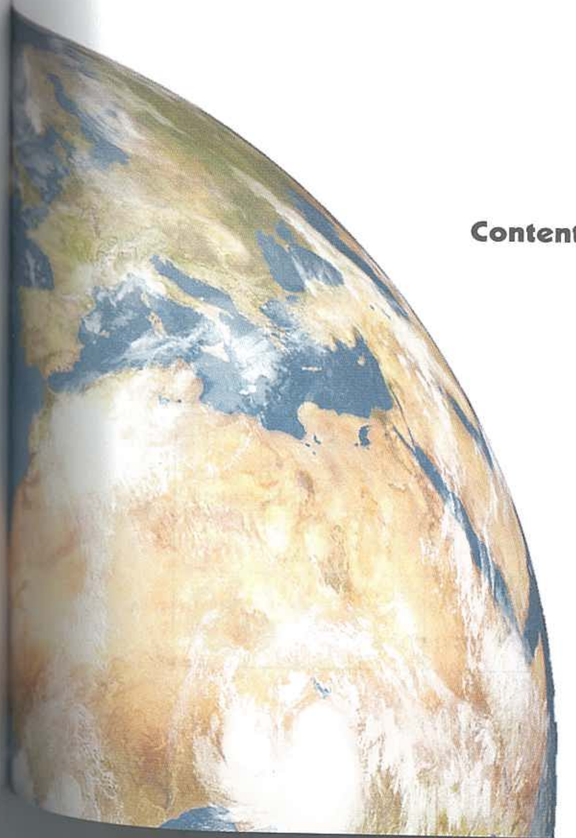
8 If a sphere has a surface area of 1000 cm^2, find to 1 decimal place:

a its radius **b** its volume.



Chapter 13

Coordinate geometry



Contents:

HISTORICAL NOTE

Sir Isaac Newton is recognised by many as one of the great mathematicians of all time. His achievements are remarkable considering mathematics, physics and astronomy were enjoyable pastimes after his main interests of chemistry, theology and alchemy.

Despite his obvious abilities, there is a story from Newton's childhood which indicates that even the greatest thinkers can find silly solutions for simple problems. He was asked to go out and cut a hole in the bottom of the barn door for the cats to go in and out. He decided to cut two holes: one for the cat and a smaller one for the kittens.

After completing school, Newton was initially made to work on a farm, but when his uncle discovered his enthusiasm for mathematics it was decided that he should attend Cambridge University.

Newton's contribution to the field of coordinate geometry included the introduction of negative values for coordinates. In his *Method of Fluxions*, Newton suggested eight new types of coordinate systems, one of which we know today as **polar coordinates**.

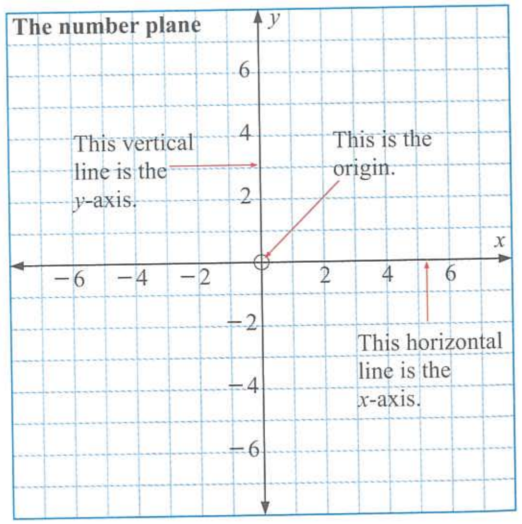
However, **René Descartes** is credited with developing the x-y coordinate system for locating points that we know as the **number plane** or **Cartesian plane**.

THE CARTESIAN PLANE

One way to specify the exact position of a point on the **two-dimensional number plane** is to use an **ordered pair** of numbers in the form (x, y).

We start with a point of reference O called the **origin**. Through it we draw two fixed lines, which are perpendicular to each other. They are a horizontal line called the **x-axis** and a vertical line called the **y-axis**.

The **x-axis** is an ordinary number line with positive numbers to the right of O and negative numbers to the left of O. Similarly, on the **y-axis** we have positives above O and negative numbers below O.



A PLOTTING POINTS

To help us identify particular points we often refer to them using a capital letter. For example, consider the points A(1, 3), B(4, −3) and C(−4, −2).

To plot the point A(1, 3):

- start at the origin O
- move right along the x-axis 1 unit
- then move upwards 3 units.

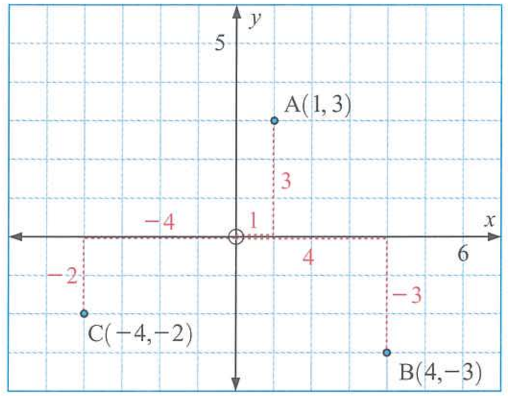
We say that 1 is the **x-coordinate** of A
and 3 is the **y-coordinate** of A.

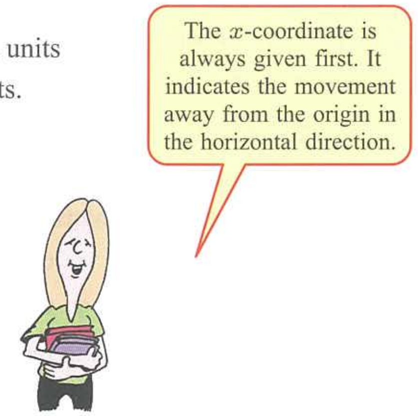
To plot the point B(4, −3):

- start at the origin O
- move right along the x-axis 4 units
- then move downwards 3 units.

To plot the point C(−4, −2):

- start at the origin O
- move left along the x-axis 4 units
- then move downwards 2 units.

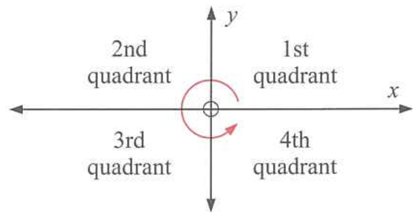




DEMO

QUADRANTS

The x and y-axes divide the Cartesian plane into four regions referred to as **quadrants**. These quadrants are numbered in an **anti-clockwise direction** as shown alongside.



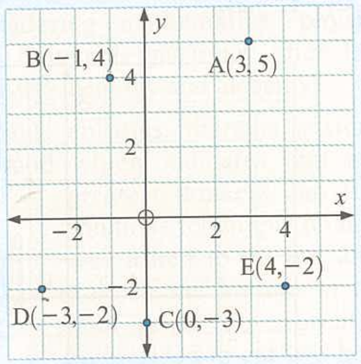
DISCUSSION — LOCATING PLACES

How can we locate the position of:

- a town or city in some country like Argentina
- a street in Auckland, New Zealand
- a point on an A4 sheet of paper?

Example 1 **Self Tutor**

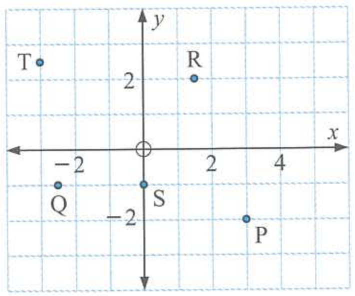
Plot the points A(3, 5), B(−1, 4), C(0, −3), D(−3, −2) and E(4, −2) on the same set of axes.



Start at O and move horizontally, then vertically.
→ is positive
← is negative
↑ is positive
↓ is negative.

EXERCISE 13A

1 State the coordinates of the points P, Q, R, S and T:



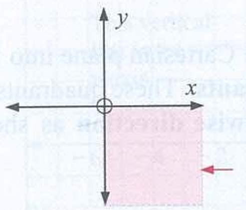
2 On the same set of axes plot the following points:

a A(3, 4) **b** B(4, 3) **c** C(2, 6) **d** D(−3, 2)
e E(5, −3) **f** F(−2, −4) **g** G(0, 5) **h** H(−3, 0)

3 State the quadrant in which each of the points in question **2** lies.

Example 2 **Self Tutor**

On a Cartesian plane, show all the points with positive x-coordinate and negative y-coordinate.



This shaded region contains all points where x is positive and y is negative.
The points on the axes are not included.

4 On different sets of axes show all points with:

a x-coordinate equal to 0
b y-coordinate equal to 0
c x-coordinate equal to 3
d y-coordinate equal to -2
e positive x-coordinate
f negative y-coordinate
g negative x and y-coordinates
h negative x and positive y-coordinates

5 On separate axes plot the following sets of points:

a $\{(0, 0), (1, 2), (2, 4), (3, 6), (4, 8)\}$ **b** $\{(0, 5), (1, 4), (2, 3), (3, 2), (4, 1)\}$

i Are the points collinear?
ii Do any of the following rules fit the set of points?

A $y = x + 1$ **B** $y = x + 5$ **C** $y = 5 - x$
D $y = 2x - 1$ **E** $y = 2x$

B LINEAR RELATIONSHIPS

Consider the pattern:

We can construct a table of values which connects the diagram number n to the number of dots P.

n	1	2	3	4
P	3	5	7	9

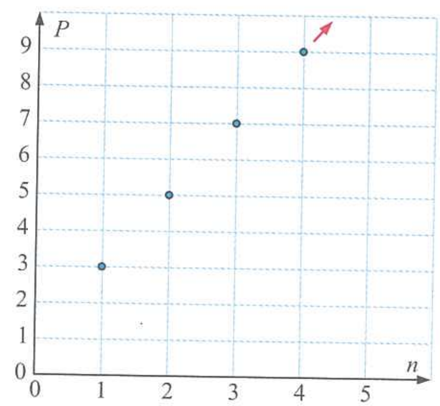
We see that to go from one diagram to the next we need to add *two more* dots.

The **equation** which connects n and P in this case is $P = 2n + 1$.

This equation is easily checked by substituting $n = 1, 2, 3, 4, ...$ to find P.

For example, if $n = 3$ then $P = 2 \times 3 + 1 = 6 + 1 = 7$ ✓

Since the number of dots P *depends* on the diagram number n, we say that n is the **independent variable** and P is the **dependent variable**.



Note:

- This is an example of a **linear** relationship because the points form a straight line. We say the points are **collinear**.
- If $n = -1$, $P = 2(-1) + 1$
$= -1$
But $(-1, -1)$ is meaningless here because we can't have a -1th diagram.
- If $n = 2\frac{1}{2}$, $P = 2(2\frac{1}{2}) + 1$
$= 6$
which is meaningless because we can't have a $2\frac{1}{2}$th diagram.
For this reason we do not connect the points by a straight line.

The independent variable is placed on the horizontal axis.

Example 3 **Self Tutor**

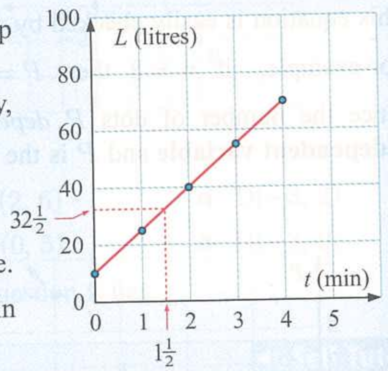
Each time when Max has only 10 litres of fuel left in his car's petrol tank, he fills it at at the petrol station. Petrol runs into the tank at 15 litres per minute. The petrol tank can only hold 70 litres.

- **a** Identify the independent and dependent variables.
- **b** Make a table of values for the number of litres L of petrol in the tank after time t minutes and plot the graph of L against t.
- **c** Is the relationship between L and t linear?
- **d** Is it sensible to join the points graphed with a straight line?
- **e** For every time increase of 1 minute, what is the change in L?
- **f** In filling the tank: **i** what amount is fixed **ii** what amount is variable?
- **g** Find the number of litres of petrol in the tank after $1\frac{1}{2}$ minutes.

a The *number of litres of petrol* in the tank depends on the *time* it has been filling. $\therefore$ *time* is the independent variable and the *number of litres of petrol* is the dependent variable.

b Each 1 minute the tank is filling adds another 15 litres of petrol.

t (min)	0	1	2	3	4
L (litres)	10	25	40	55	70



c The points lie in a straight line so the relationship is linear.

d Yes, as we could add petrol for $2\frac{1}{2}$ minutes, say, or put 56 litres of petrol in the tank.

e an increase of 15 litres

f **i** The fixed amount is 10 litres.
ii The variable amount is 15 litres per minute.

g After $1\frac{1}{2}$ minutes there are $32\frac{1}{2}$ litres of petrol in the tank.

EXERCISE 13B

1 Each week a travel agent receives a basic salary of €200. In addition the agent is paid €30 for each ticket sold to Europe.

- **a** What are the independent and dependent variables?
- **b** Construct a table and draw a graph of income I against tickets sold t where $t = 0$, 1, 2, 3, ..., 10.
- **c** Is the relationship linear?
- **d** Is it sensible to join the points with a straight line?
- **e** For each ticket sold, what will be the increase in income?
- **f** **i** What is the fixed income? **ii** What is the variable income?

2 One litre cartons of 'Long-life' milk can be bought for 85 pence each.

- **a** Copy and complete the table:

Number of cartons n	0	1	2	3	4	5	6	7	8
Cost in pounds £C									

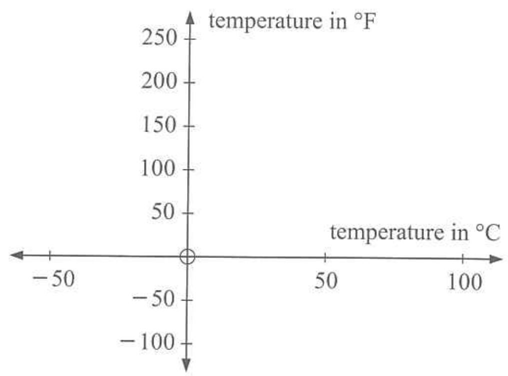
- **b** Plot the graph of C against n.
- **c** Identify the independent and dependent variables.
- **d** Is the relationship between C and n linear?
- **e** Is it sensible to join the points graphed with a straight line?
- **f** For each extra carton of milk bought, what is the change in C?
- **g** Find the cost of 4 cartons of milk.
- **h** How many cartons of milk could be bought for £5.95?

3 Jason has a large container for water. It already contains 4 litres when he starts to fill it using a 1.5 litre jug.

- **a** Make a table of values for the volume of water (V litres) in the container after Jason has emptied n full jugs of water ($n = 0, 1, 2, ..., 8$) into it.
- **b** What are the independent and dependent variables?
- **c** Plot the graph of V against n.
- **d** Is the relationship between V and n linear?
- **e** Is it sensible to join the points graphed with a straight line?
- **f** For each full jug of water added, what is the change in V?
- **g** What volume of water is in the container after Jason has emptied 5 full jugs into it?
- **h** How many full jugs must be emptied into the container to give a volume of 14.5 litres in the container?

4 Susan is planning a holiday in Alaska. It is important that she knows what temperatures to expect so that she can pack appropriate clothing. She looks at the international weather report and sees that the temperatures are given in degrees Fahrenheit (°F) as used in the USA, whereas she is familiar with degrees Celsius (°C) as used in New Zealand.

- **a** Draw a set of axes as shown, using a scale of 1 cm represents 10°C on the horizontal axis and 1 cm represents 20°F on the vertical axis.



- **b** Identify the independent and dependent variables.
- **c** There is a linear relationship between °F and °C. The boiling point of water is 100°C or 212°F. The freezing point of water is 0°C or 32°F.

Mark these points on your graph and join them with a straight line.

d Find the point where the number of degrees Celsius equals the same number of degrees Fahrenheit. What is the temperature?

e Susan sees that the maximum temperatures were:

i 46°F on Monday **ii** 36°F on Tuesday.

Convert these temperatures to °C.

f Use your graph to complete the following chart:

Temperature °F					20	10	0
Temperature °C	10	20	30	40			

C PLOTTING LINEAR GRAPHS

Consider the equation $y = \frac{1}{2}x - 1$ which describes the relationship between two variables x and y.

For any given value of x, we can use the equation to find the value of y. These values form the coordinates (x, y) of a point on the graph of the equation.

The value of y depends on the value of x, so the independent variable is x and the dependent variable is y.

When $x = -2$, $y = \frac{1}{2}(-2) - 1 = -1 - 1 = -2$

When $x = 2$, $y = \frac{1}{2}(2) - 1 = 1 - 1 = 0$

From calculations like these we construct a **table of values**:

x	-3	-2	-1	0	1	2	3
y	$-2\frac{1}{2}$	-2	$-1\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$

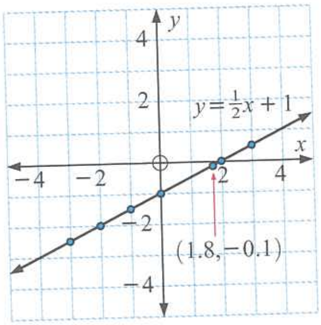
So, the points $(-3, -2\frac{1}{2})$, $(-2, -2)$, $(-1, -1\frac{1}{2})$, $(0, -1)$, ... all satisfy $y = \frac{1}{2}x - 1$ and lie on its graph.

Notice also that if $x = 1.8$ then $y = \frac{1}{2}(1.8) - 1 = 0.9 - 1 = -0.1$

$\therefore$ $(1.8, -0.1)$ also satisfy $y = \frac{1}{2}x - 1$.

In fact there are infinitely many points which satisfy $y = \frac{1}{2}x - 1$.

These points make up a **continuous** straight line which continues indefinitely in both directions. We indicate this using arrowheads.



Example 4 **Self Tutor**

Consider the equation $y = x - 2$.

a What are the independent and dependent variables?

b Construct a table of values using $x = -3, -2, -1, 0, 1, 2$ and 3.

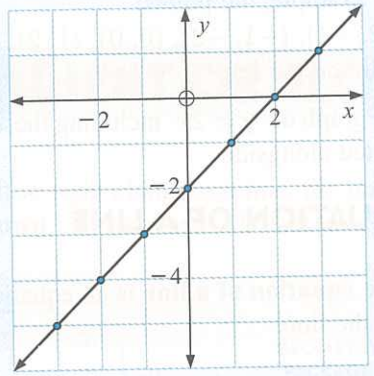
c Draw the graph of $y = x - 2$.

a x is the independent variable. y is the dependent variable.

b

x	-3	-2	-1	0	1	2	3
y	-5	-4	-3	-2	-1	0	1

c



EXERCISE 13C

1 For the following equations:

i State the independent and dependent variables.

ii Construct a table of values using $x = -3, -2, -1, 0, 1, 2, 3$.

iii Plot the graph.

a $y = x$ **b** $y = 2x$ **c** $y = \frac{1}{2}x$ **d** $y = -2x$

e $y = 2x + 3$ **f** $y = -2x + 3$ **g** $y = \frac{1}{4}x + 1$ **h** $y = -\frac{1}{4}x + 1$

2 Arrange the graphs of $y = x$, $y = 2x$ and $y = \frac{1}{2}x$ in order of steepness. What part of the equation do you suspect controls the degree of steepness of a line?

3 Compare the graphs of $y = 2x$ and $y = -2x$. What part of the equation do you suspect controls whether the graph slopes forwards or backwards?

4 Compare the graphs of $y = 2x$, $y = 2x + 3$ and $y = \frac{1}{4}x + 1$. What part of the equation do you suspect controls where the graph cuts the y-axis?

DISCUSSION

How many points lie on the line with equation $y = 2x + 3$?

Are there gaps between the points on the line?

Does the line have finite length?

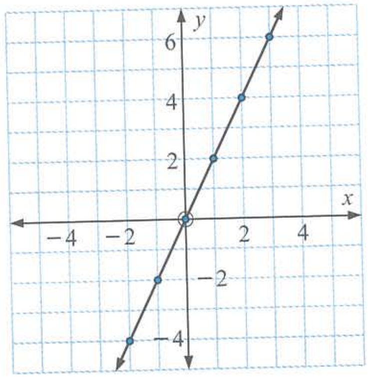
D THE EQUATION OF A LINE

Consider the line with equation $y = 2x$.

For all points on the line, the y-coordinate is always double the x-coordinate.

For example, the points:
$(-2, -4)$, $(-1, -2)$, $(0, 0)$, $(1, 2)$, $(2, 4)$ and $(3, 6)$ all lie on the line.

The graph of $y = 2x$ including these points is shown plotted alongside.



EQUATION OF A LINE

The **equation of a line** is an equation which connects the x and y-coordinates of **all** points on the line.

Example 5 Self Tutor

State in words the meaning of the equation $y = 3x + 7$.

$y = 3x + 7$ connects the x and y values for every point on the line where the y-coordinate is three times the x-coordinate, plus 7.

Example 6 Self Tutor

By inspection only, find the equations of the straight lines passing through the following points:

a

x	1	2	3	4	5
y	-3	-6	-9	-12	-15

b

x	1	2	3	4	5
y	5	4	3	2	1

a Each y-value is -3 times its corresponding x-value, so $y = -3x$.
b The sum of each x and y-value is always 6, so $x + y = 6$.

EXERCISE 13D

1 State in words the meaning of the equation:
a $y = x + 3$ **b** $y = 5x$ **c** $y = 2x - 6$ **d** $x + y = 5$

2 By inspection only, find the equations of the straight lines passing through the following points:

a

x	1	2	3	4	5
y	4	8	12	16	20

b

x	2	3	4	5	6
y	4	5	6	7	8

c

x	0	1	2	3
y	0	-2	-4	-6

d

x	0	1	2	3
y	3	2	1	0

e

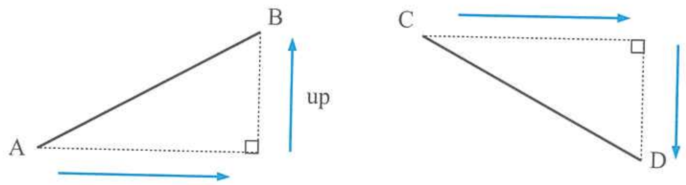
x	1	2	3	4	5
y	3	5	7	9	11

f

x	1	2	3	4	5
y	1	3	5	7	9

E GRADIENT OR SLOPE

Consider the two diagrams below. As we go from left to right along each line, the line from A to B travels **up**, but the line from C to D travels **down**. We say that [AB] has an upwards **gradient** whereas [CD] has a downwards gradient.

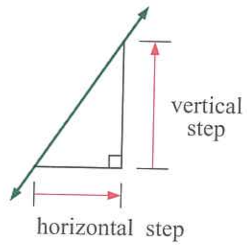


GEOMETRY PACKAGE

We use **gradient** or **slope** as a measure of a line's steepness. Horizontal lines are said to have **zero gradient**. Upwards sloping lines have positive gradients, whereas downwards sloping lines have negative gradients.

Gradient is the comparison of the vertical step to the horizontal step.

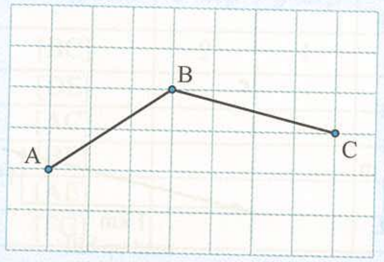
$$\textbf{gradient} = \frac{\textbf{vertical step}}{\textbf{horizontal step}}$$



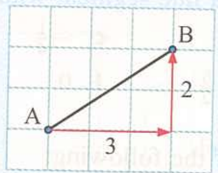
When calculating the gradient of a line, it is a good idea to first move horizontally in the positive direction.

Example 7 Self Tutor

Find the gradient of:
a [AB] **b** [BC]

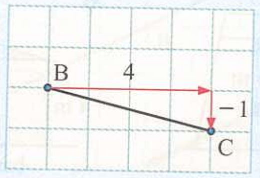


a



gradient $= \frac{2}{3}$

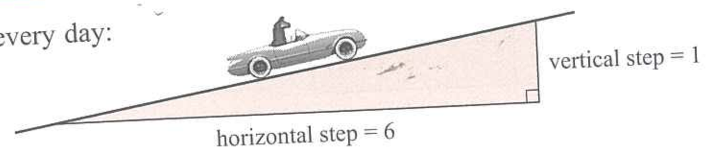
b

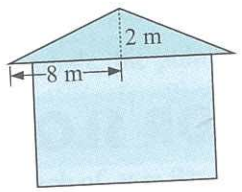


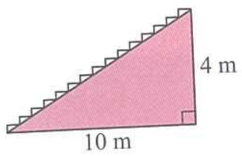
gradient $= \frac{-1}{4}$
$= -\frac{1}{4}$

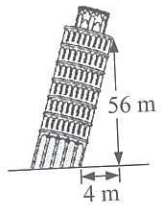
We see examples of gradient every day:

gradient of road $= \frac{1}{6}$









gradient of roof $= \frac{2}{8}$
$= \frac{1}{4}$

gradient of stairs $= \frac{4}{10}$
$= \frac{2}{5}$

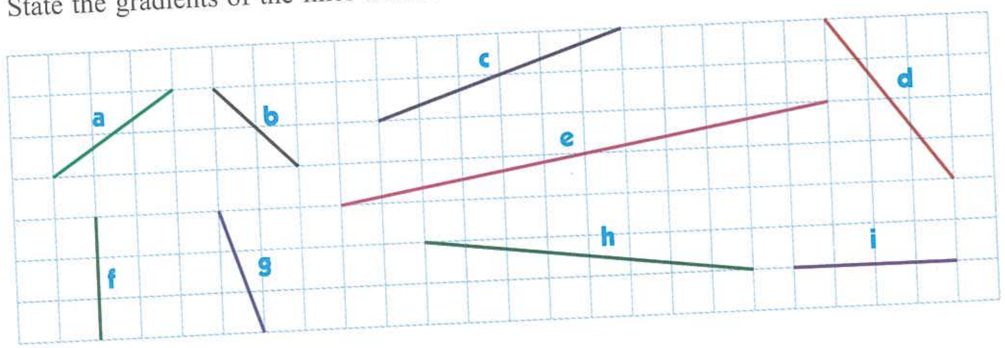
gradient of tower $= \frac{56}{4}$
$= 14$

For a **horizontal line** the vertical step is 0. This gives 0 as the numerator, so the **gradient of a horizontal line is 0**.

For a **vertical line**, the horizontal step is 0. So the denominator is 0 and the **gradient of the vertical line is undefined**.

EXERCISE 13E.1

1 State the gradients of the lines labelled:

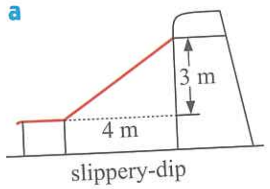


2 On grid paper draw a line segment with gradient:

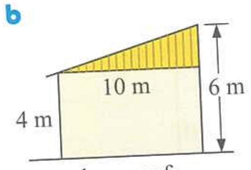
a $\frac{1}{2}$	**b** $\frac{2}{3}$	**c** $-\frac{1}{4}$	**d** $2 = \frac{2}{1}$	**e** 1	**f** -3
g $\frac{4}{3}$	**h** $-\frac{5}{2}$	**i** 0	**j** $1\frac{1}{5}$	**k** 10	**l** undefined

3 Find the gradients of the following:

a



b



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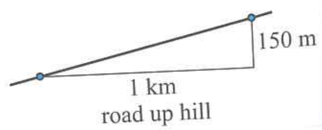
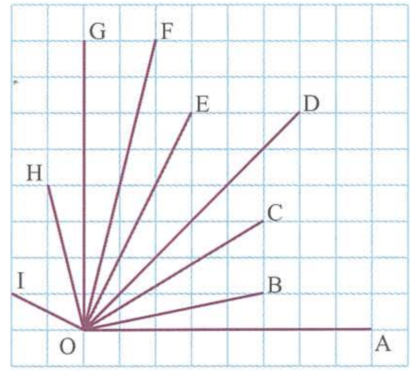


4 **a** Determine the gradient of:

i [OA]	**ii** [OB]	**iii** [OC]
iv [OD]	**v** [OE]	**vi** [OF]
vii [OG]	**viii** [OH]	**ix** [OI]

b Using the results of **a**, copy and complete the following statements:

- **i** The gradient of a horizontal line is ......
- **ii** The gradient of a vertical line is ......
- **iii** As line segments become steeper, their gradients ......



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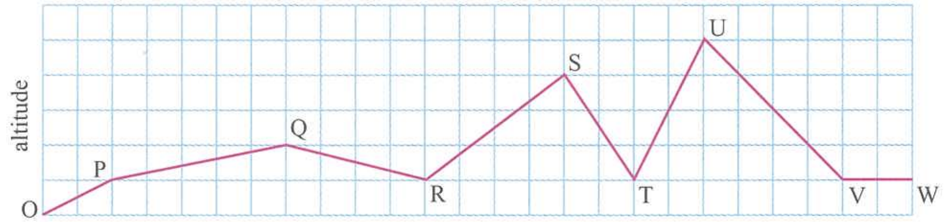


Imagine that you are walking across the countryside from O to W, i.e., from left to right.

- **a** Indicate when you are going uphill.
- **b** Indicate when you are going downhill.
- **c** Where is the steepest positive gradient?
- **d** Where is the steepest negative gradient?
- **e** Where is the gradient 0?
- **f** Where is the gradient positive and least?

6 By plotting the points on graph paper, find the gradients of the lines joining the following pairs of points:

a O(0, 0) and A(2, 6)	**b** O(0, 0) and B(−4, 2)
c G(0, −1) and H(2, 5)	**d** K(1, 1) and L(−2, −2)
e M(3, 1) and N(−1, 3)	**f** P(−2, 4) and Q(2, 0)

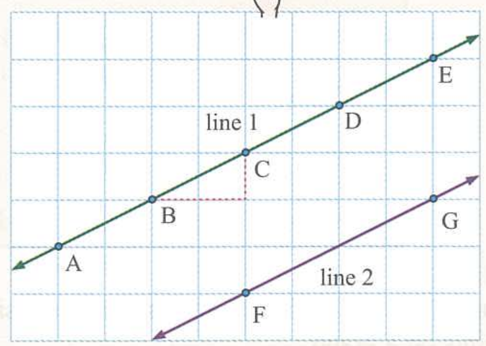
INVESTIGATION 1 — THE GRADIENT OF A LINE

What to do:

WORKSHEET

1 Copy and complete:

Line segment	*x-step*	*y-step*	$\frac{y\text{-step}}{x\text{-step}}$
[BC]	2	1	$\frac{1}{2}$
[DE]			
[AC]			
[BE]			
[AE]			
[FG]			



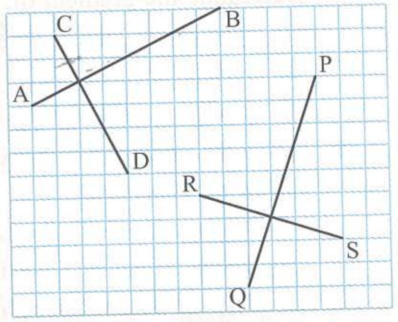
2 Copy and complete:

a Lines [AB] and [CD] are ......

b Lines [PQ] and [RS] are ......

c gradient of [AB] = ...
gradient of [PQ] = ...
gradient of [CD] = ...
gradient of [RS] = ...

d gradient of [AB] × gradient of [CD] = ...
gradient of [PQ] × gradient of [RS] = ...



3 Copy and complete:

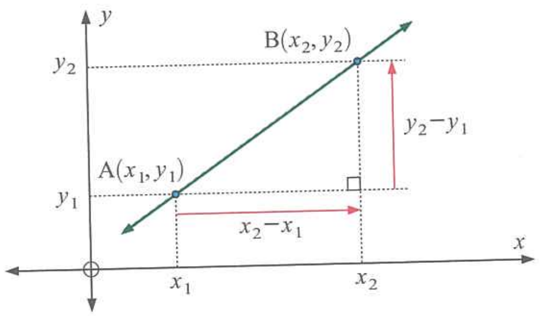
- The gradient of a straight line is ......
- Parallel lines have ......
- Perpendicular lines have gradients whose product is ......

THE GRADIENT FORMULA

Although we can find gradients easily using steps on a diagram, it is often quicker to use a formula.

For points $A(x_1, y_1)$ and $B(x_2, y_2)$, the vertical step is $y_2 - y_1$ and the horizontal step is $x_2 - x_1$.

$\therefore$ the gradient is $\dfrac{y_2 - y_1}{x_2 - x_1}$.



So, the **gradient** of the line through (x_1, y_1) and (x_2, y_2) is $\dfrac{y_2 - y_1}{x_2 - x_1}$.

Example 8 **Self Tutor**

Find the gradient of [PQ] for $P(1, 3)$ and $Q(4, -2)$.

$P(1, 3)$ with $x_1 = 1$, $y_1 = 3$; $Q(4, -2)$ with $x_2 = 4$, $y_2 = -2$.

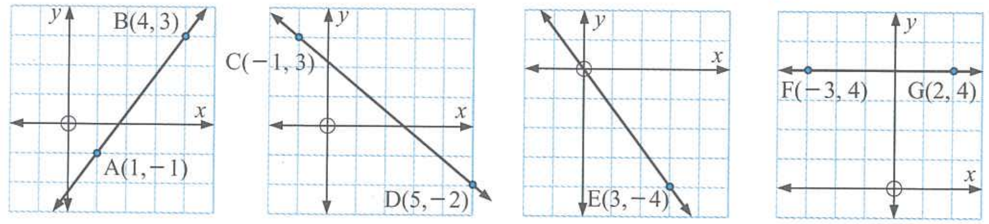
$$\text{gradient of [PQ]} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2-3}{4-1} = \frac{-5}{3} = -\frac{5}{3}$$

The advantage of finding the gradient by using a formula is that we do not have to plot the points on a graph.

EXERCISE 13E.2

1 For the following graphs, find the gradients by:

i considering horizontal and vertical steps **ii** using the gradient formula.



2 Use the gradient formula to find the gradients of the lines joining the following pairs of points:

a O(0, 0) and A(2, 6) **b** O(0, 0) and B(−4, 2)

c O(0, 0) and C(2, −12) **d** O(0, 0) and D(1, −5)

e E(1, 0) and F(1, 5) **f** G(0, −1) and H(2, −1)

g I(1, 1) and J(3, 3) **h** K(1, 1) and L(−2, −2)

i M(3, 1) and N(−1, 3) **j** P(−2, 4) and Q(2, 0)

3 Which method do you think is easier to use to find the gradient of the line joining R(−50, 460) and S(115, 55)? Explain your answer.

4 Suppose a straight line makes an angle θ^o with the positive x-axis. Copy and complete:

a The gradient of any horizontal line is ......

b The gradient of any vertical line is ......

c **i** If $\theta = 0^o$, the gradient is ...... **ii** If $\theta = 45^o$, the gradient is ......

iii If $\theta = 90^o$, the gradient is ...... **iv** If θ is acute, the gradient is ......

v If θ is obtuse, the gradient is ......

F GRAPHING LINES FROM EQUATIONS

INVESTIGATION 2 GRAPHS OF THE FORM $y = mx + c$

The use of a **graphics calculator** or suitable **graphing package** is recommended for this investigation.

GRAPHING PACKAGE

What to do:

1 On the same set of axes, graph the family of lines of the form $y = mx$:

a where $m = 1, 2, 3, \frac{1}{2}, \frac{1}{3}$ **b** where $m = -1, -2, -3, -\frac{1}{2}, -\frac{1}{3}$

2 Find the gradients of the lines in **1**.

3 Describe the effect of m in the equation $y = mx$.

4 On the same set of axes, graph the family of lines of the form $y = 2x + c$ where $c = 0, 2, 4, -1, -3$.

5 Describe the effect of c in the equation $y = 2x + c$.

The point where a straight line meets the x-axis is called its **x-intercept**.
The point where a straight line meets the y-axis is called its **y-intercept**.

From **Investigation 2** you should have discovered that:

$y = mx + c$ is the equation of a straight line with gradient m and y-intercept c.

Example 9 — Self Tutor

Give the gradient and y-intercept of the line with equation:

a $y = 3x - 2$ **b** $y = 7 - 2x$ **c** $y = 0$

a $y = 3x - 2$ has $m = 3$ and $c = -2$
$\therefore$ gradient is 3 and y-intercept is -2

b $y = 7 - 2x$
or $y = -2x + 7$ has $m = -2$ and $c = 7$
$\therefore$ gradient is -2 and y-intercept is 7

c $y = 0$
or $y = 0x + 0$ has $m = 0$ and $c = 0$
$\therefore$ gradient is 0 and y-intercept is 0

EXERCISE 13F.1

1 Write down the equation of a line with:

a gradient 2 and y-intercept 11
b gradient 4 and y-intercept -6
c gradient -3 and y-intercept $\frac{1}{2}$
d gradient -1 and y-intercept 4
e gradient 0 and y-intercept 7
f gradient $\frac{2}{3}$ and y-intercept 6.

2 Give the **i** gradient **ii** y-intercept of the line with equation:

a $y = 4x + 8$
b $y = -3x + 2$
c $y = 6 - x$
d $y = -2x + 3$
e $y = -2$
f $y = 11 - 3x$
g $y = \frac{1}{2}x - 5$
h $y = 3 - \frac{3}{2}x$
i $y = \frac{2}{5}x + \frac{4}{5}$
j $y = \dfrac{x+1}{2}$
k $y = \dfrac{2x - 10}{5}$
l $y = \dfrac{11 - 3x}{2}$

GRAPHING $y = mx + c$

To draw the graph of $y = mx + c$ we could do one of the following:

- construct a table of values using at least 3 points and then graph the points
- use the y-intercept and gradient
- use the x and y-intercepts.

GRAPHING USING y-INTERCEPT AND GRADIENT

To draw the graph of $y = mx + c$ using the y-intercept and gradient:

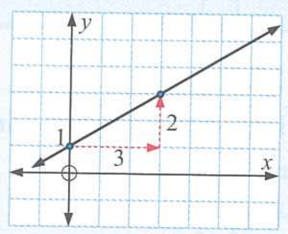
- find m and c
- locate the y-intercept and plot this point
- from the gradient, use the x and y-steps to locate another point
- join these two points and extend the line in either direction.

Example 10 — Self Tutor

Use the y-intercept and gradient of $y = \frac{2}{3}x + 1$ to graph the line.

For $y = \frac{2}{3}x + 1$, $m = \frac{2}{3}$ and $c = 1$
$\therefore$ the y-intercept is 1

and the gradient is $\dfrac{2}{3}$ ← y-step / ← x-step

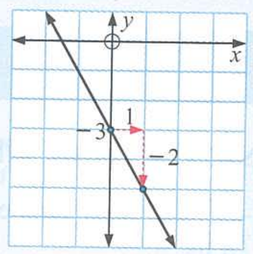


Example 11 — Self Tutor

Use the y-intercept and gradient of $y = -2x - 3$ to graph the line.

For $y = -2x - 3$, $m = -2$
and $c = -3$
$\therefore$ the y-intercept is -3

and $m = \dfrac{-2}{1}$ ← y-step / ← x-step



EXERCISE 13F.2

1 For each of the following:
i find the gradient and y-intercept **ii** sketch the graph.

a $y = x + 3$
b $y = -x + 4$
c $y = 2x + 2$
d $y = \frac{1}{2}x - 1$
e $y = 3x$
f $y = -\frac{1}{2}x$
g $y = -2x + 1$
h $y = 3 - \frac{1}{3}x$
i $y = 3$

GRAPHING USING THE AXES INTERCEPTS

The y-intercept gives us one point to plot.

The x-intercept can be found by letting $y = 0$ in the equation of the line and then solving for x.

Example 12 — Self Tutor

Use axes intercepts to graph the line with equation $y = 2x - 5$.

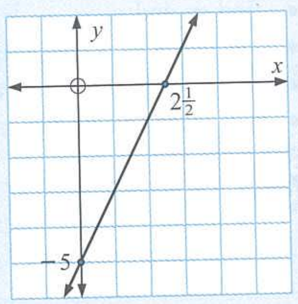
The y-intercept is -5.

When $y = 0$, $2x - 5 = 0$

$\therefore\ 2x = 5$

$\therefore\ x = \frac{5}{2}$

So, the x-intercept is $2\frac{1}{2}$.



EXERCISE 13F.3

1 Use axes intercepts to graph the line with equation:

- a $y = x + 4$
- b $y = -x + 3$
- c $y = 2x + 4$
- d $y = -2x + 5$
- e $y = \frac{1}{2}x - 2$
- f $y = 3 - \frac{1}{2}x$
- g $y = \frac{1}{3}x + 1$
- h $y = 2 - \frac{x}{3}$
- i $y = \frac{x+2}{3}$

2 Can lines of the form $y = mx$ be graphed using intercepts only? How could you graph these lines?

G OTHER LINE FORMS

HORIZONTAL AND VERTICAL LINES

Lines parallel to the x-axis and lines parallel to the y-axis are special cases which we need to treat with care.

INVESTIGATION 3 — **HORIZONTAL AND VERTICAL LINES**

What to do:

1 Using graph paper, plot the following sets of points on the Cartesian plane. Rule a line through each set of points.

- a (5, 2), (5, 4), (5, −2), (5, −4), (5, 0)

- b (−1, 2), (−1, 4), (−1, −2), (−1, −4), (−1, 0)
- c (0, 2), (0, 4), (0, −2), (0, −4), (0, 0)
- d (−1, 3), (2, 3), (0, 3), (4, 3), (−3, 3)
- e (0, −4), (1, −4), (4, −4), (−2, −4), (3, −4)
- f (1, 0), (0, 0), (5, 0), (−1, 0), (−2, 0)

2 Can you state the gradient of each line? If so, what is it?

3 Can you state the y-intercept of each line? If so, what is it?

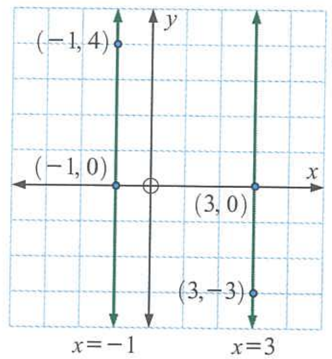
4 How are these lines different from other lines previously studied?

5 Can you state the equation of the line?

VERTICAL LINES

The graph alongside shows the vertical lines $x = -1$ and $x = 3$.

For all points on a vertical line, regardless of the value of the y-coordinate, the value of the x-coordinate is the same.



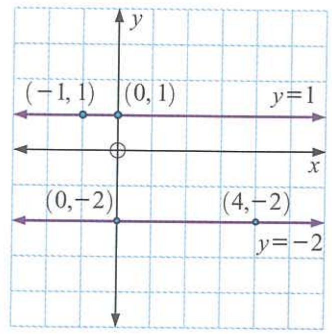
All **vertical** lines have equations of the form $x = a$.

The gradient of a vertical line is **undefined**.

HORIZONTAL LINES

The graph alongside shows the horizontal lines $y = -2$ and $y = 1$.

For all points on a horizontal line, regardless of the value of the x-coordinate, the value of the y-coordinate is the same.



All **horizontal** lines have equations of the form $y = c$.

The gradient of a horizontal line is **zero**.

EXERCISE 13G.1

1 Identify as either a vertical or horizontal line and hence plot the graphs of:

a $y = 3$ **b** $x = 4$ **c** $y = -2$ **d** $x = -6$

2 Identify as either a vertical or horizontal line:

a a line with zero gradient **b** a line with undefined gradient.

3 Find the equation of:

a the x-axis **b** the y-axis

c a line parallel to the x-axis and two units above it

d a line parallel to the y-axis and 3 units to the left of it.

THE GENERAL FORM

Equations of the form $Ax + By = C$ where A, B and C are constants are also **linear**. This form is called the **general form** of the equation of a line.

For example, $2x - 3y = -6$ is the equation of a straight line in general form. In this case $A = 2$, $B = -3$ and $C = -6$.

We can rearrange equations given in general form into the $y = mx + c$ form and hence find their gradient.

Example 13 🔊 **Self Tutor**

Make y the subject of the equation and hence find the gradient of:

a $x - y = 4$ **b** $2x + 3y = -6$

A line in the form $y = mx + c$ has gradient m.

a
$$x - y = 4$$
$$\therefore\ x - y - x = 4 - x \quad \{\text{subtracting } x \text{ from both sides}\}$$
$$\therefore\ -y = 4 - x \quad \{\text{simplifying}\}$$
$$\therefore\ \frac{-y}{-1} = \frac{4}{-1} - \frac{x}{-1} \quad \{\text{dividing each term by } -1\}$$
$$\therefore\ y = -4 + x \quad \{\text{simplifying}\}$$
$$\therefore\ y = x - 4 \quad \text{and so the gradient is 1.}$$

b
$$2x + 3y = -6$$
$$\therefore\ 2x + 3y - 2x = -6 - 2x \quad \{\text{subtracting } 2x \text{ from both sides}\}$$
$$\therefore\ 3y = -6 - 2x \quad \{\text{simplifying}\}$$
$$\therefore\ \frac{3y}{3} = \frac{-6}{3} - \frac{2x}{3} \quad \{\text{dividing both sides by 3}\}$$
$$\therefore\ y = -2 - \tfrac{2}{3}x$$
$$\therefore\ y = -\tfrac{2}{3}x - 2 \quad \text{and so the gradient is } -\tfrac{2}{3}.$$

DISCUSSION

We can graph equations given in general form by converting them into the $y = mx + c$ form and then using the gradient and y-intercept.
Is there a better or quicker method of graphing equations in the general form?

EXERCISE 13G.2

1 Find the gradient of the line with equation:

a $y = 2x - 3$ **b** $y = -3x + 4$ **c** $y = \dfrac{2x}{3} + 5$

d $y = \dfrac{3x+1}{4}$ **e** $y = 5 - 2x$ **f** $y = \frac{3}{5} - x$

2 Make y the subject of the equation and hence find the gradient of:

a $x + y = 5$ **b** $2x + y = 4$ **c** $y - 3x = 5$

d $2x - 3y = 6$ **e** $2y + x - 6 = 0$ **f** $6 - 2y + 5x = 0$

g $3x + 7y = -21$ **h** $2x - 5y = 10$ **i** $5x + 3y = 30$

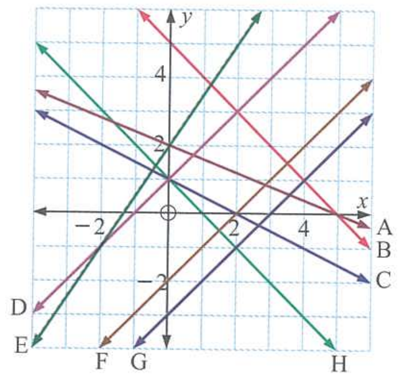
3 Match each equation to one of the graphs by using y-intercepts and gradients:

a $y = x + 1$ **b** $y = 1 - x$

c $2y = 3x + 4$ **d** $x + y = 5$

e $2x + 5y = 10$ **f** $4 - 2x = 4y$

g $x - y = 3$ **h** $\frac{1}{2}y = \frac{1}{2}x - 1$



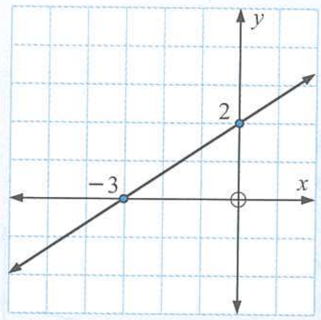
Example 14 🔊 **Self Tutor**

Use axes intercepts to draw the graph of the line $2x - 3y = -6$.

Points on the y-axis have an x-coordinate of zero.

The line cuts the y-axis when $x = 0$
$$\therefore\ -3y = -6$$
$$\therefore\ \frac{-3y}{-3} = \frac{-6}{-3}$$
$$\therefore\ y = 2$$

The line cuts the x-axis when $y = 0$
$$\therefore\ 2x = -6$$
$$\therefore\ \frac{2x}{2} = \frac{-6}{2}$$
$$\therefore\ x = -3$$



4 Use axes intercepts to draw the graph of:

a $x + y = 6$ **b** $2x + y = 4$ **c** $3x - y = 5$

d $2x + 3y = 6$ **e** $3x - 4y = 12$ **f** $x + 3y = -6$

g $2x - 5y = 10$ **h** $2x + 7y = 14$ **i** $3x - 4y = 8$

H FINDING EQUATIONS FROM GRAPHS

To determine the equation of a line we need to know its gradient and at least one other point on it.

If we are given two points on a graph then we can find its gradient m by using $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.

So, if we know the y-intercept and one other point then we can find the **equation** connecting x and y.

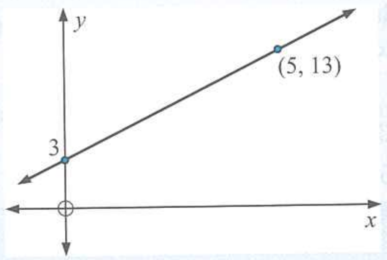
Example 15 Self Tutor

For the graph alongside, determine:

a the gradient of the line

b the y-intercept

c the equation of the line.



a (0, 3) and (5, 13) lie on the line

$\therefore$ the gradient $m = \dfrac{13-3}{5-0}$

$= \frac{10}{5}$

$= 2$

b The y-intercept is 3.

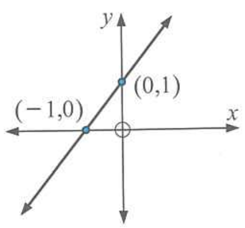
c The equation of the line is $y = 2x + 3$.

EXERCISE 13H

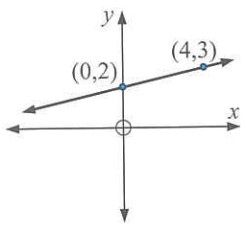
1 For each of the following lines, find:

i the gradient **ii** the y-intercept **iii** the equation of the line.

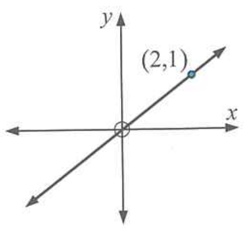
a



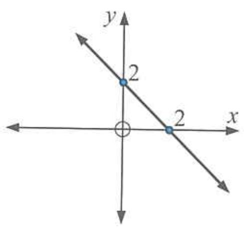
b



c



d



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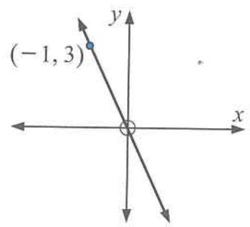

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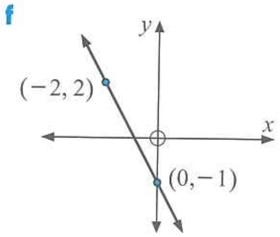
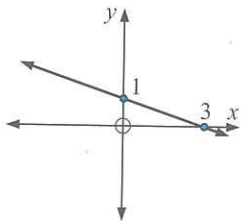


g



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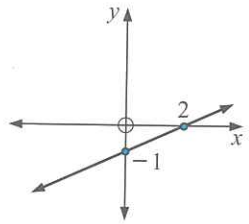
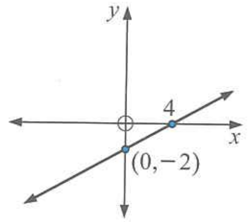
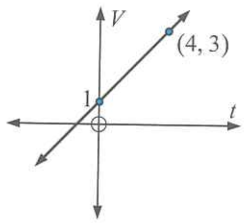


i



2 Find the rule connecting the variables in:

a



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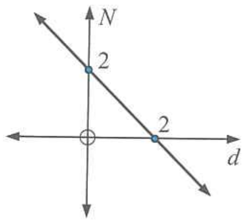
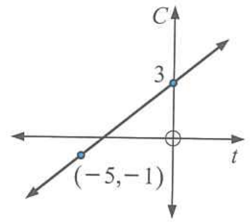


c



I POINTS ON LINES

Once we have found the equation of a line, we can easily check to see whether a given point lies on it.

We simply replace x by the x-coordinate of the point being tested and see if the equation produces the y-coordinate.

Example 16 Self Tutor

Determine whether (2, 11) and (−3, −3) lie on the line with equation $y = 3x + 5$.

If $x = 2$, $y = 3 \times 2 + 5$

$= 6 + 5$

$= 11$

$\therefore$ (2, 11) lies on the line.

If $x = -3$, $y = 3 \times (-3) + 5$

$= -9 + 5$

$= -4 \neq -3$

$\therefore$ (−3, −3) does not lie on the line.

EXERCISE 13I

1 Determine whether the given point satisfies the given equation, or in other words whether the point lies on the graph of the line:

- **a** $(2, 5)$, $y = 4x - 3$
- **b** $(3, -1)$, $y = 2x - 7$
- **c** $(-4, 2)$, $y = -x + 3$
- **d** $(-1, 7)$, $y = -2x + 5$
- **e** $(4, -1)$, $y = -2x + 7$
- **f** $(1, 6)$, $y = -3x + 5$

INVESTIGATION 4 THE GRAPH OF $y = \sqrt{x}$

This investigation is best attempted using a **graphing package** or **graphics calculator**.

GRAPHING PACKAGE

What to do:

1 **a** Draw the graph of $y = \sqrt{x}$ from a table of values.
b Use technology to check your graph.
c Explain why there is no graph to the left of the vertical axis.
d Explain why no part of the graph is below the x-axis.

2 **a** On the same set of axes, graph $y = \sqrt{x}$ and $y = -\sqrt{x}$.
b What is the geometrical relationship between the two graphs?

3 **a** Draw the graphs of $y = x^2$ and $y = \sqrt{x}$ on the same set of axes.
b What is the geometrical relationship between the two graphs?

4 **a** On the same set of axes graph $y = \sqrt{x}$, $y = \sqrt{x} + 3$ and $y = \sqrt{x} - 3$.
b What is the geometrical relationship between the three graphs?
c Do the graphs have the same basic shape?

5 **a** On the same set of axes plot the graphs of $y = \sqrt{x}$, $y = \sqrt{x-3}$ and $y = \sqrt{x+3}$.
b Do the graphs have the same basic shape?
c What is the geometrical relationship between the three graphs?

LINKS click here

HOW FAR IS IT FROM YOUR CHIN TO YOUR FINGERTIPS?

Areas of interaction:
Approaches to learning

REVIEW SET 13A

1 On a set of axes show all points with an x-coordinate equal to -2.

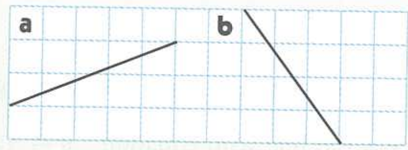
2 A lawyer charges a €75 consultation fee and then €120 per hour thereafter.
a Identify the dependent and independent variables.
b Make a table of values for the cost €C of an appointment with the lawyer for time t hours where $t = 0, 1, 2, 3, 4$.
c Is the relationship between C and t linear?
d If you graphed this relationship, is it sensible to join the points graphed with a straight line? Give a reason for your answer.
e For every increase of 1 hour for t, what is the change in C?
f For an appointment with the lawyer, what is:
i the fixed cost **ii** the variable cost?

3 For the equation $y = -2x + 3$:
a Construct a table of values with $x = -3, -2, -1, 0, 1, 2, 3$.
b Plot the graph.
c Find: **i** the y-intercept **ii** the x-intercept.

4 Find, by inspection, the equation of the straight line passing through the points:

x	0	1	2	3	4	5
y	3	2	1	0	-1	-2

5 State the gradients of the given lines:



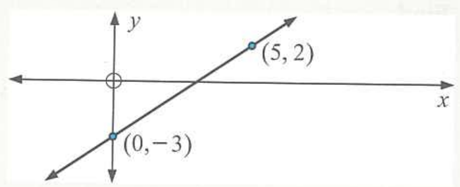
6 Use the gradient formula to find the gradient of the straight line through the points A$(-2, 3)$ and B$(1, -4)$.

7 Write down the equation of the line with gradient 5 and y-intercept -2.

8 For the line with equation $y = -\frac{1}{3}x + 2$:
a find the gradient and the y-intercept, and hence
b sketch the graph.

9 Use axes intercepts to draw the graph of $2x - 3y = 6$.

10 For the line alongside, find:
a the gradient
b the y-intercept
c the equation of the line.



11 Determine whether the point $(-2, 5)$ lies on the line with equation $y = -2x + 1$.

REVIEW SET 13B

1 Illustrate on a set of axes the region where both x and y are negative.

2 A tank contains 400 litres of water. The tap is left on and 25 litres of water escape per minute. V is the volume of water remaining in the tank t minutes after the tap is switched on.

- **a** What are the dependent and independent variables?
- **b** Make a table of values for V and t.
- **c** Is the relationship between V and t linear?
- **d** If you graphed this relationship, is it sensible to join the points graphed with a straight line? Give a reason for your answer.
- **e** For every increase of 1 minute for t, what is the change in V?

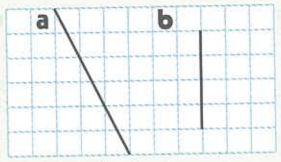
3 For the equation $y = 3x - 2$:

- **a** Construct a table of values with $x = -2, -1, 0, 1, 2$.
- **b** Plot the graph.
- **c** Find: **i** the y-intercept **ii** the x-intercept.

4 Find, by inspection, the equation of the straight line passing through the points:

x	0	1	2	3	4
y	0	-3	-6	-9	-12

5 State the gradients of the given lines:



6 On grid paper, draw a line with a gradient of: **a** $-\frac{4}{3}$ **b** 0.

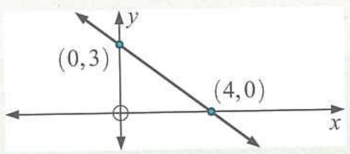
7 For the line with equation $y = \frac{3}{4}x + 2$:

- **a** find the gradient and the y-intercept, and hence
- **b** sketch the graph.

8 Find the gradient of the line with equation $5x - 2y = 12$.

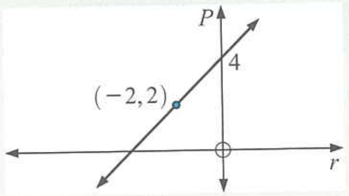
9 For the line alongside, find:

- **a** the gradient
- **b** the y-intercept
- **c** the equation of the line.



10 Find:

- **a** the gradient of the line shown
- **b** the equation connecting the variables.



11 Determine whether the point (1, 2) lies on the line with equation $y = 3x - 5$.

Chapter 14

Simultaneous equations

Contents: